ISSN 2394-9716

International Journal of Novel Research in Interdisciplinary Studies
Vol. 11, Issue 6, pp: (25-31), Month: November – December 2024, Available at: www.noveltyjournals.com

Study of Four Types of Matrix Fractional Integrals

Chii-Huei Yu

School of Mathematics and Statistics,
Zhaoqing University, Guangdong, China

DOI: https://doi.org/10.5281/zenodo.14135998

Published Date: 13-November-2024

***Abstract*: In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional integral and a new multiplication of fractional analytic functions, we study four types of matrix fractional integrals. Using some methods, we can evaluate these four types of matrix fractional integrals. Moreover, our results are generalizations of classical calculus results.**

***Keywords*: Jumarie's modified R-L fractional integral, new multiplication, fractional analytic functions, matrix fractional integrals.**

I. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis, involving the research and applications of arbitrary order integrals and derivatives. Fractional calculus originated from a problem put forward by L'Hospital and Leibniz in 1695. Therefore, the history of fractional calculus was formed more than 300 years ago, and fractional calculus and classical calculus have almost the same long history. Since then, fractional calculus has attracted the attention of many contemporary great mathematicians, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, and H. Weyl. With the efforts of researchers, the theory of fractional calculus and its applications have developed rapidly. On the other hand, fractional calculus has wide applications in physics, mechanics, electrical engineering, viscoelasticity, biology, control theory, dynamics, economics, and other fields [1-16].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [17-21]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional integral and a new multiplication of fractional analytic functions, we study the following four types of matrix fractional integrals:

$$\left({}_{0}I_{x}^{\alpha}\right)\left[cos_{\alpha}\left(cosh_{\alpha}(rAx^{\alpha})\right)\right],$$

$$\left({}_{0}I_{x}^{\alpha}\right)\left[cos_{\alpha}\left(sinh_{\alpha}(rAx^{\alpha})\right)\right],$$

$$\left({}_{0}I_{x}^{\alpha}\right)\left[sin_{\alpha}\left(cosh_{\alpha}(rAx^{\alpha})\right)\right],$$

$$\left({}_{0}I_{x}^{\alpha}\right)\left[sin_{\alpha}\left(sinh_{\alpha}(rAx^{\alpha})\right)\right],$$

where $0 < \alpha \leq 1$, r is a real number, A is a real matrix, and A is invertible. Using some methods, we can evaluate these four types of matrix fractional integrals. In fact, our results are generalizations of classical calculus results.

International Journal of Novel Research in Interdisciplinary Studies
Vol. 11, Issue 6, pp: (25-31), Month: November – December 2024, Available at: www.noveltyjournals.com

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([22])**:** Let $0 < \alpha \le 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([23])**:** *If* α, β, x_0, c *are real numbers and* $\beta \ge \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c] = 0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([24])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([25])**:** Let $0 < \alpha \le 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ & = \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ & = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([26])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([27])**:** If $0 < \alpha \leq 1$, x is a real number, and A is a matrix. Then the matrix α-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes n}. \tag{13}$$

And the matrix α-fractional cosine and matrix α-fractional sine function are defined as follows:

$$cos_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^{2n} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{14}$$

and

$$sin_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^{2n+1} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{15}$$

In addition, the matrix α-fractional hyperbolic cosine and hyperbolic sine function are defined as follows:

$$cosh_\alpha(Ax^\alpha) = \frac{1}{2}[E_\alpha(Ax^\alpha) + E_\alpha(-Ax^\alpha)] = \sum_{n=0}^{\infty} A^{2n} \frac{x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{16}$$

and

$$sinh_\alpha(Ax^\alpha) = \frac{1}{2}[E_\alpha(Ax^\alpha) - E_\alpha(-Ax^\alpha)] = \sum_{n=0}^{\infty} A^{2n+1} \frac{x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{17}$$

Definition 2.7 ([27])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$.

Theorem 2.8 (fractional binomial theorem)**:** *If $0 < \alpha \leq 1$, m is a positive integer and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then*

$$[f_\alpha(x^\alpha) + g_\alpha(x^\alpha)]^{\otimes_\alpha m} = \sum_{k=0}^{n} \binom{m}{k} \big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha (m-k)} \otimes_\alpha \big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha k}, \tag{18}$$

where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$.

III. MAIN RESULTS

In this section, we evaluate four types of matrix fractional integrals. At first, a lemma is needed.

Lemma 3.1: *If $0 < \alpha \leq 1$, m is a non-negative integer, r is a real number, and A is a real matrix, then*

$$[cosh_\alpha(rAx^\alpha)]^{\otimes_\alpha m} = \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} [E_\alpha((m-2k)rAx^\alpha)], \tag{19}$$

International Journal of Novel Research in Interdisciplinary Studies
Vol. 11, Issue 6, pp: (25-31), Month: November – December 2024, Available at: www.noveltyjournals.com

$$[sinh_\alpha(rAx^\alpha)]^{\otimes_\alpha m} = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^k[E_\alpha((m-2k)rAx^\alpha)]. \quad (20)$$

Proof $[cosh_\alpha(rAx^\alpha)]^{\otimes_\alpha m}$

$$= \left[\frac{1}{2}[E_\alpha(rAx^\alpha) + E_\alpha(-rAx^\alpha)]\right]^{\otimes_\alpha m}$$

$$= \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}[E_\alpha(rAx^\alpha)]^{\otimes_\alpha (m-k)} \otimes_\alpha [E_\alpha(-rAx^\alpha)]^{\otimes_\alpha k} \quad \text{(by fractional binomial theorem)}$$

$$= \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}[E_\alpha((m-2k)rAx^\alpha)] .$$

And

$$[sinh_\alpha(rAx^\alpha)]^{\otimes_\alpha m}$$

$$= \left[\frac{1}{2}[E_\alpha(rAx^\alpha) - E_\alpha(-rAx^\alpha)]\right]^{\otimes_\alpha m}$$

$$= \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}[E_\alpha(rAx^\alpha)]^{\otimes_\alpha (m-k)} \otimes_\alpha [-E_\alpha(-rAx^\alpha)]^{\otimes_\alpha k} \quad \text{(by fractional binomial theorem)}$$

$$= \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^k[E_\alpha((m-2k)rAx^\alpha)] . \quad \text{q.e.d.}$$

Theorem 3.2: *If* $0 < \alpha \leq 1$, r *is a real number*, $r \neq 0$, *and* A *is a real invertible matrix, then*

$$\left({}_0I_x^\alpha\right)\left[cos_\alpha\left(rAcosh_\alpha(x^\alpha)\right)\right]$$

$$= I \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2 \cdot 2^{2n}} + \frac{1}{r}A^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)! \cdot 2^{2n}}\sum_{\substack{k=0\\k\neq n}}^{2n}\binom{2n}{k}\frac{1}{2n-2k}E_\alpha((2n-2k)rAx^\alpha), \quad (21)$$

$$\left({}_0I_x^\alpha\right)\left[cos_\alpha\left(sinh_\alpha(rAx^\alpha)\right)\right]$$

$$= I \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha \cdot \sum_{n=0}^{\infty}\frac{1}{(n!)^2 \cdot 2^{2n}} + \frac{1}{r}A^{-1} \cdot \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)! \cdot 2^{2n}}\sum_{\substack{k=0\\k\neq n}}^{2n}\binom{2n}{k}(-1)^k\frac{1}{2n-2k}E_\alpha((2n-2k)rAx^\alpha), \quad (22)$$

$$\left({}_0I_x^\alpha\right)\left[sin_\alpha\left(cosh_\alpha(rAx^\alpha)\right)\right]$$

$$= I \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha \cdot \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}}\binom{2n+1}{\frac{n+1}{2}} + \frac{1}{r}A^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}}\sum_{\substack{k=0\\k\neq\frac{n+1}{2}}}^{2n}\binom{2n+1}{k}\frac{1}{2n+1-2k}E_\alpha((2n+1-2k)rAx\alpha,$$

(23)

$$\left({}_0I_x^\alpha\right)\left[sin_\alpha\left(sinh_\alpha(rAx^\alpha)\right)\right]$$

$$= I \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha \cdot \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}}\binom{2n+1}{\frac{n+1}{2}}(-1)^{\frac{n+1}{2}} + \frac{1}{r}A^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}}\sum_{\substack{k=0\\k\neq\frac{n+1}{2}}}^{2n}\binom{2n+1}{k}(-1)^k\frac{1}{2n+1-2k}E_\alpha((2n+1-2k)rAx\alpha.$$

(24)

Proof $\left({}_0I_x^\alpha\right)\left[cos_\alpha\left(cosh_\alpha(rAx^\alpha)\right)\right]$

$$= \left({}_0I_x^\alpha\right)\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(cosh_\alpha(rAx^\alpha)\right)^{\otimes_\alpha 2n}\right]$$

International Journal of Novel Research in Interdisciplinary Studies
Vol. 11, Issue 6, pp: (25-31), Month: November – December 2024, Available at: www.noveltyjournals.com

$= \left({}_0I_x^{\alpha} \right) \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} [E_{\alpha}((2n-2k)rAx^{\alpha})] \right]$ (by Lemma 3.1)

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} \left({}_0I_x^{\alpha} \right) [E_{\alpha}((2n-2k)rAx^{\alpha})]$

$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \binom{2n}{n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \sum_{\substack{k=0 \\ k \neq n}}^{2n} \binom{2n}{k} \left({}_0I_x^{\alpha} \right) [E_{\alpha}((2n-2k)rAx^{\alpha})]$

$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2 \cdot 2^{2n}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \sum_{\substack{k=0 \\ k \neq n}}^{2n} \binom{2n}{k} \frac{1}{2n-2k} \frac{1}{r} A^{-1} E_{\alpha}((2n-2k)rAx^{\alpha})$

$= I \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2 \cdot 2^{2n}} + \frac{1}{r} A^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \sum_{\substack{k=0 \\ k \neq n}}^{2n} \binom{2n}{k} \frac{1}{2n-2k} E_{\alpha}((2n-2k)rAx^{\alpha}).$

$\left({}_0I_x^{\alpha} \right) [cos_{\alpha}(sinh_{\alpha}(rAx^{\alpha}))]$

$= \left({}_0I_x^{\alpha} \right) \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(sinh_{\alpha}(rAx^{\alpha}) \right)^{\otimes_{\alpha} 2n} \right]$

$= \left({}_0I_x^{\alpha} \right) \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k [E_{\alpha}((2n-2k)rAx^{\alpha})] \right]$ (by Lemma 3.1)

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \left({}_0I_x^{\alpha} \right) [E_{\alpha}((2n-2k)rAx^{\alpha})]$

$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \binom{2n}{n} (-1)^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \sum_{\substack{k=0 \\ k \neq n}}^{2n} \binom{2n}{k} (-1)^k \left({}_0I_x^{\alpha} \right) [E_{\alpha}((2n-2k)rAx^{\alpha})]$

$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{1}{(n!)^2 \cdot 2^{2n}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \sum_{\substack{k=0 \\ k \neq n}}^{2n} \binom{2n}{k} (-1)^k \left({}_0I_x^{\alpha} \right) [E_{\alpha}((2n-2k)rAx^{\alpha})]$

$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{1}{(n!)^2 \cdot 2^{2n}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \sum_{\substack{k=0 \\ k \neq n}}^{2n} \binom{2n}{k} (-1)^k \frac{1}{2n-2k} \cdot \frac{1}{r} A^{-1} E_{\alpha}((2n-2k)rAx^{\alpha})$

$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{1}{(n!)^2 \cdot 2^{2n}} + \frac{1}{r} A^{-1} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2^{2n}} \sum_{\substack{k=0 \\ k \neq n}}^{2n} \binom{2n}{k} (-1)^k \frac{1}{2n-2k} E_{\alpha}((2n-2k)rAx^{\alpha}).$

$\left({}_0I_x^{\alpha} \right) [sin_{\alpha}(cosh_{\alpha}(rAx^{\alpha}))]$

$= \left({}_0I_x^{\alpha} \right) \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(cosh_{\alpha}(rAx^{\alpha}) \right)^{\otimes_{\alpha} (2n+1)} \right]$

$= \left({}_0I_x^{\alpha} \right) \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} [E_{\alpha}((2n+1-2k)rAx^{\alpha})] \right]$ (by Lemma 3.1)

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \left({}_0I_x^{\alpha} \right) [E_{\alpha}((2n+1-2k)rAx^{\alpha})]$

$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \binom{2n+1}{\frac{n+1}{2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \sum_{\substack{k=0 \\ k \neq \frac{n+1}{2}}}^{2n} \binom{2n+1}{k} \left({}_0I_x^{\alpha} \right) [E_{\alpha}((2n+1-2k)rAx^{\alpha})]$

$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \binom{2n+1}{\frac{n+1}{2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \sum_{\substack{k=0 \\ k \neq \frac{n+1}{2}}}^{2n} \binom{2n+1}{k} \frac{1}{2n+1-2k} \frac{1}{r} A^{-1} E_{\alpha}((2n+1-$

$2k)rAx\alpha$

$$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \binom{2n+1}{\frac{n+1}{2}} + \frac{1}{r} A^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \sum_{\substack{k=0 \\ k \neq \frac{n+1}{2}}}^{2n} \binom{2n+1}{k} \frac{1}{2n+1-2k} E_{\alpha}((2n+1-2k)rAx^{\alpha}).$$

$$\left({}_0 I_x^{\alpha}\right)\left[\sin_{\alpha}\left(\sinh_{\alpha}(rAx^{\alpha})\right)\right]$$

$$= \left({}_0 I_x^{\alpha}\right)\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\sinh_{\alpha}(rAx^{\alpha})\right)^{\otimes_{\alpha}(2n+1)}\right]$$

$$= \left({}_0 I_x^{\alpha}\right)\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k [E_{\alpha}((2n+1-2k)rAx^{\alpha})]\right] \quad \text{(by Lemma 3.1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k \left({}_0 I_x^{\alpha}\right)[E_{\alpha}((2n+1-2k)rAx^{\alpha})]$$

$$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \binom{2n+1}{\frac{n+1}{2}} (-1)^{\frac{n+1}{2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \sum_{\substack{k=0 \\ k \neq \frac{n+1}{2}}}^{2n} \binom{2n+1}{k} (-1)^k \left({}_0 I_x^{\alpha}\right)[E_{\alpha}((2n+1-2k)rAx^{\alpha})]$$

$$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \binom{2n+1}{\frac{n+1}{2}} (-1)^{\frac{n+1}{2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \sum_{\substack{k=0 \\ k \neq \frac{n+1}{2}}}^{2n} \binom{2n+1}{k} (-1)^k \frac{1}{2n+1-2k} \frac{1}{r} A^{-1} E_{\alpha}((2n+1-2k)rAx^{\alpha}).$$

$$= I \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \binom{2n+1}{\frac{n+1}{2}} (-1)^{\frac{n+1}{2}} + \frac{1}{r} A^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 2^{2n+1}} \sum_{\substack{k=0 \\ k \neq \frac{n+1}{2}}}^{2n} \binom{2n+1}{k} (-1)^k \frac{1}{2n+1-2k} E_{\alpha}((2n+1-2k)rAx^{\alpha}).$$

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional integral and a new multiplication of fractional analytic functions, we solve four types of matrix fractional integrals by using some techniques. Furthermore, our results are generalizations of ordinary calculus results. In the future, we will continue to use our methods to solve the problems in fractional differential equations and engineering mathematics.

REFERENCES

[1] A. Carpinteri, F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York, 1997.

[2] J. Sabatier, O. P. Agrawal, J.A. Tenreiro Machado, Advances in Fractional Calculus. Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht, 2007.

[3] V. E. Tarasov, Review of Some Promising Fractional Physical Models, International Journal of Modern Physics. vol. 27, no. 9, 2013.

[4] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp. 41-45, 2016.

[5] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[6] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics vol.23, pp. 397-404, 2002.

[7] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, Fractals and Fractional Calculus in Continuum Mechanics, pp. 291-348, Springer, Wien, Germany, 1997.

[8] R. Magin, Fractional calculus in bioengineering, part 1, Critical Reviews in Biomedical Engineering, vol. 32, no,1. pp.1-104, 2004.

[9] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[10] H. A. Fallahgoul, S. M. Focardi and F. J. Fabozzi, Fractional calculus and fractional processes with applications to financial economics, theory and application, Elsevier Science and Technology, 2016.

[11] M. F. Silva, J. A. T. Machado, and I. S. Jesus, Modelling and simulation of walking robots with 3 dof legs, in Proceedings of the 25th IASTED International Conference on Modelling, Identification and Control (MIC '06), pp. 271-276, Lanzarote, Spain, 2006.

[12] M. Teodor, Atanacković, Stevan Pilipović, Bogoljub Stanković, Dušan Zorica, Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes, John Wiley & Sons, Inc., 2014.

[13] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[14] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[15] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[16] F. Duarte and J. A. T. Machado, Chaotic phenomena and fractional-order dynamics in the trajectory control of redundant manipulators, Nonlinear Dynamics, vol. 29, no. 1-4, pp. 315-342, 2002.

[17] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[18] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, Inc., 1974.

[19] K. S. Miller and B. Ross, An introduction to the Fractional Calculus and Fractional Differential Equations, A Wiley-Interscience Publication, John Wiley & Sons, New York, USA, 1993.

[20] S. Das, Functional Fractional Calculus for System Identification and Control, 2nd ed., Springer-Verlag, Berlin, 2011.

[21] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[22] C. -H, Yu, Studying three types of matrix fractional integrals, International Journal of Interdisciplinary Research and Innovations, vol. 12, no. 4, pp. 35-39, 2024.

[23] C. -H, Yu, Evaluating fractional derivatives of two matrix fractional functions based on Jumarie type of Riemann-Liouville fractional derivative, International Journal of Engineering Research and Reviews, vol. 12, no. 4, pp. 39-43, 2024.

[24] C. -H, Yu, Study of fractional Fourier series expansions of two types of matrix fractional functions, International Journal of Mathematics and Physical Sciences Research, vol. 12, no. 2, pp. 13-17, 2024.

[25] C. -H, Yu, Fractional partial differential problem of some matrix two-variables fractional functions, International Journal of Mechanical and Industrial Technology, vol. 12, no. 2, pp. 6-13, 2024.

[26] C. -H, Yu, Study of two matrix fractional integrals by using differentiation under fractional integral sign, International Journal of Civil and Structural Engineering Research, vol. 12, no. 2, pp. 24-30, 2024.

[27] C. -H, Yu, Fractional differential problem of two matrix fractional hyperbolic functions, International Journal of Recent Research in Civil and Mechanical Engineering, vol. 11, no. 2, pp. 1-4, 2024.